Lorentz-like covariant equations of non-relativistic fluids

2003 J. Phys. A: Math. Gen. 36 2009

(http://iopscience.iop.org/0305-4470/36/8/301)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 2009–2026 PII: S0305-4470(03)52573-2

Lorentz-like covariant equations of non-relativistic fluids

M de Montigny[1,2], **F C Khanna**[2,3] **and A E Santana**[2,4]

[1] Faculté Saint-Jean, University of Alberta, 8406-91 Street, Edmonton, Alberta, Canada T6C 4G9
[2] Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2J1
[3] TRIUMF, 4004, Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3
[4] Instituto de Física, Universidade Federal da Bahia, Campus de Ondina, Salvador, Bahia, Brazil 40210-340

E-mail: montigny@phys.ualberta.ca, khanna@phys.ualberta.ca and santana@ufba.br

Received 20 August 2002
Published 12 February 2003
Online at stacks.iop.org/JPhysA/36/2009

Abstract

We use a geometrical formalism of Galilean invariance to build various hydrodynamics models. It consists in embedding the Newtonian spacetime into a non-Euclidean $4+1$ space and provides thereby a procedure that unifies models otherwise apparently unrelated. After expressing the Navier–Stokes equation within this framework, we show that slight modifications of its Lagrangian allow us to recover the Chaplygin equation of state as well as models of superfluids for liquid helium (with both its irrotational and rotational components). Other fluid equations are also expressed in a covariant form.

PACS numbers: 11.10.Ef, 05.60.Gg, 11.30.−j, 45.20.Jj, 67.40.Pm

1. Introduction

This paper is the continuation of [1] in which we have used a five-dimensional formulation of Galilean covariance to recover a model for compressible irrotational barotropic fluids with pressure quadratic in mass density (see [2]). Our approach follows [2–4] in which non-relativistic classical and quantum fields are utilized with the Galilei invariance as a guide for constructing many-body theories. The central tool thereupon is the Galilean tensor calculus in five dimensions, investigated more systematically in [5]. This allows one to treat the Galilean invariance in a way that looks like the Lorentz transformation; the Lorentz covariance being most useful in relativistic quantum field theory where it serves as a guiding principle in writing down action functionals. However, this is not quite the approach followed in [2–4]; the authors leaving aside the geometrical structure to define as soon as possible the physical $3+1$ spacetime variables. Here we work a bit further within five dimensions by writing our equations in a covariant form and only then performing the embedding of the Newtonian spacetime into this five-dimensional space. Although not investigated any further here, an interesting step would be to solve the equations altogether in the five-dimensional space and then define the embedding.

To our knowledge a manifestly covariant formalism of the Galilean invariance in five dimensions first appeared in [6]. Then it was shown, among other things, that the Galilean tensor calculus with a non-singular metric follows naturally as a limit of the Lorentz covariance. A similar five-dimensional formalism was mentioned in the textbook [7]. It was then investigated more formally in [8] in the framework of gravitational models where the geometric structure consists of a five-dimensional Lorentz metric with a covariantly constant vector field which is spacelike in the Lorentz case and null in the Galilean case. The usual four-dimensional Newtonian spacetime occurs as the quotient manifold of the orbits of this vector field. As discussed in section 3 this framework has been used recently in fluid models to explain the appearance of Galilean, Poincaré and conformal symmetries in the description of membranes by (2 + 1)-dimensional field theory or, as was shown to be equivalent, irrotational isentropic fluid motion for specific values of the potential (see [9–17]). An alternative method, similar to ours, to relate such systems to the Nambu–Goto action of a membrane in higher dimensions is given in [14]. More fluid models are investigated in [16, 17]. Our purpose is to take this covariant-like prescription of the Galilean invariance as a guide to construct many-body theories, as suggested by Takahashi in [2, 3]. Hereafter we recover various models of fluids: some are well known (Navier–Stokes, nonlinear Schrödinger) whereas some are more exotic (Takahashi model for liquid helium II, Chaplygin equation of state).

Regarding eventual experimental applications, particularly in condensed matter physics, we consider the present approach as providing a prescription for building hydrodynamical models for fluids and superfluids. The latter is interesting, given the recent realizations of Bose–Einstein condensation in a series of experiments on alkali vapours (rubidium, sodium and lithium) [18] as well as hydrogen [19], and especially the evidence that these vapours behave like superfluids [20]. Excitations associated with Bose–Einstein condensation have been thoroughly studied in relation to ^{4}He and the investigations of collective excitations therein provide a good test of the various finite-temperature many-body theories of interacting Bose gases. We will mention such an example in section 4.2. For a review on Bose–Einstein condensation see [21].

The five-dimensional approach to Galilean covariance used in this paper is based on a space such that a Galilean boost acts on the extended spacetime represented by a 5-vector

$$x = (x^1, x^2, x^3, x^4, x^5) = (\mathbf{x}, t, s) \tag{1}$$

as

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{V}t \\ t' &= t \\ s' &= s - \mathbf{V}\cdot\mathbf{x} + \tfrac{1}{2}\mathbf{V}^2 t \end{aligned} \tag{2}$$

with relative velocity $\mathbf{V}$. The dimensions of the components are

$$\begin{aligned} [\mathbf{x}] &= L \qquad \text{(length)} \\ [t] &= T \qquad \text{(time)} \\ [s] &= \frac{L^2}{T}. \end{aligned} \tag{3}$$

The scalar product $(A|B) = A^\mu B_\mu \equiv \mathbf{A}\cdot\mathbf{B} - A_4 B_5 - A_5 B_4$ of two 5-vectors A and B is invariant under the transformations given in equation (2), which suggests basing the

Galilean tensor calculus on the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4}$$

In this paper we refer to this metric as the *Galilean metric*. The underlying vector space is clearly a 4 + 1 de Sitter space with metric diag(+ + + − +) since with the change of basis:

$$\mathbf{y} \equiv \mathbf{x} \qquad y^4 \equiv \frac{1}{\sqrt{2}}(x_4 + x_5) \qquad y^5 \equiv \frac{1}{\sqrt{2}}(x_4 - x_5) \tag{5}$$

the scalar product of a 5-vector with itself now reads $\mathbf{y}^2 - (y^4)^2 + (y^5)^2$. Group-theoretically speaking, the 11-dimensional extended Galilei algebra arises as a subalgebra of the 15-dimensional Poincaré algebra of this 4 + 1 manifold. Obviously the usual ten-dimensional Poincaré algebra of 3 + 1 spacetime is also a subalgebra of this enlarged Poincaré algebra [5, 8]. In [14, 16] a similar five-dimensional approach has been used to explain the existence of typically *relativistic* symmetries (Poincaré, conformal) in *non-relativistic* systems such as the one discussed in section 3. This suggests that a similar occurrence of surprising symmetries might appear in theories constructed at once within our framework. From the relation mentioned at the beginning of this paragraph, it is not surprising that this approach is fruitful in explaining why some Poincaré generators can intervene for non-relativistic theories, such as the Chaplygin gas model [14].

The introduction of an additional parameter s can be understood classically in terms of quasi-invariance of the following free particle Lagrangian (see [4, 22]):

$$L_{\text{free}} = \frac{1}{2}m\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2. \tag{6}$$

Indeed, if we apply the transformation of equation (2) we get

$$\begin{aligned} L'_{\text{free}} &= \frac{1}{2}m\left(\frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t}\right)^2 \\ &= \frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{V}t)\right)^2 \\ &= \frac{1}{2}m\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(-m\mathbf{x}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2 t\right) \\ &= L_{\text{free}} + \frac{\mathrm{d}f}{\mathrm{d}t} \end{aligned} \tag{7}$$

where $f \equiv -m(\mathbf{x})\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2 t$. The Lagrangian can be made invariant by redefining it as follows:

$$L \equiv L_{\text{free}} - m\frac{\mathrm{d}s}{\mathrm{d}t} \tag{8}$$

as long as s transforms appropriately. Indeed, from the last line of equation (7) we find

$$\begin{aligned} L &= L'_{\text{free}} - m\left(\frac{\mathrm{d}s'}{\mathrm{d}t}\right) \\ &= L_{\text{free}} + \frac{\mathrm{d}f}{\mathrm{d}t} - m\left(\frac{\mathrm{d}s'}{\mathrm{d}t}\right). \end{aligned} \tag{9}$$

By taking it equal to equation (8) we get

$$\frac{1}{m}\frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}s'}{\mathrm{d}t} = -\frac{\mathrm{d}s}{\mathrm{d}t} \tag{10}$$

which shows that the extra parameter s must transform as in equation (2).

The quantum version of this argument lies in the fact that the invariance of the Schrödinger equation for a free particle of mass m,

$$-\frac{1}{2m}\nabla^2\psi(\mathbf{x},t) = \mathrm{i}\partial_t\psi(\mathbf{x},t) \tag{11}$$

cannot be achieved under the simple condition

$$\psi'(\mathbf{x}',t') = \psi(\mathbf{x},t) \tag{12}$$

because the transformed quantities then give rise to the equation

$$-\frac{1}{2m}\nabla^2\psi(\mathbf{x},t) = \mathrm{i}\partial_t\psi(\mathbf{x},t) - \mathrm{i}\mathbf{V}\cdot\nabla\psi(\mathbf{x},t). \tag{13}$$

(For the rest of this paper ∂_t, ∂_x, etc denote the partial derivatives with respect to t, x, etc.) Now the unwanted last term can be removed by introducing an extra phase into the wavefunction

$$\psi'(\mathbf{x}',t') = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}m(s-s')}\psi(\mathbf{x},t) \tag{14}$$

where $s = s(\mathbf{x}, t; \mathbf{V})$. This is the quantum counterpart of equation (8) and it follows from the relation $\psi \propto \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int L\mathrm{d}t}$. By using equations (2) for $\mathbf{x}$ and t, and (12) we find by substitution into the Schrödinger equation that

$$\nabla(m(s'-s)) = -m\mathbf{V} \qquad \partial_t(m(s'-s)) = \tfrac{1}{2}m\mathbf{V}^2 \tag{15}$$

so that equation (2) is recovered. As explained in detail in [22], this extra parameter is linked to the non-trivial central extension of the Galilei group.

The approach used in the following consists in writing the Lagrangian in a covariant form in five dimensions, and then using the fact that the Newtonian spacetime is embedded in this space. The specific embedding leaves us with a Galilei-invariant model. More details are given in [1, 5]. Throughout this paper we shall use the following embedding:

$$(\mathbf{x},t) \hookrightarrow x = (x^1,\ldots,x^5) \equiv \left(\mathbf{x}, v_4 t, \frac{s}{v_5}\right) \tag{16}$$

of the Newtonian spacetime into the five-dimensional space. The tilde notation (for $\tilde{\phi}$, $\tilde{\Phi}$, $\tilde{\mathcal{L}}$, etc) is used for objects defined in five dimensions whereas their non-tilde counterparts denote the same quantities in Newtonian spacetime. For a real field $\tilde{\phi}$ we define

$$\tilde{\phi}(x) \equiv \phi(\mathbf{x},t) + a_0 s \tag{17}$$

where a_0 is a dimensionless constant ($\tilde{\phi}$, ϕ and s all have units of $\frac{L^2}{T}$), whereas for a complex field $\tilde{\psi}$ we use the definition

$$\tilde{\psi}(x) \equiv \mathrm{e}^{\mathrm{i}a_0 s}\psi(\mathbf{x},t). \tag{18}$$

Note that the dimensionless argument of the exponential in equation (18) should be $\mathrm{i}a_0 s/\hbar$, where a_0 has units of mass (we work with $\hbar \equiv 1$). This corresponds to the definitions given in section 5 of the first article in [14]. Let us also mention that the definitions of the fields given in equations (17) are in agreement with the equivariance condition given in equation (5.1) of the first article in [14]:

$$\xi^\mu\partial_\mu\tilde{\psi} = \mathrm{i}\tilde{\psi}. \tag{19}$$

If one defines thereupon $\xi^\mu = (\mathbf{0}, 0, -1)$ and uses equation (16) then equation (19) becomes

$$-\partial_s\tilde{\psi} = \mathrm{i}\tilde{\psi} \quad \rightarrow \quad \tilde{\psi}(x) = \mathrm{e}^{-\mathrm{i}s}\psi(\mathbf{x}, t). \tag{20}$$

If one understands the field $\tilde{\phi}$ as playing the role of a phase for $\tilde{\psi}$ then one obtains equation (17) with $a_0 = -1$.

The models that we examine hereafter are all obtained with slight modifications of the Lagrangian

$$\tilde{\mathcal{L}} \propto \tilde{\Phi}^p \tag{21}$$

for some fractional number p and where

$$\tilde{\Phi} \propto \partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - V(\tilde{\phi}) \tag{22}$$

with the embeddings described previously and conditions similar to

$$v_4 = v_5 \qquad \text{and} \qquad a_0 = -1. \tag{23}$$

Using the definitions in equations (16), (17), (22) and (23) we find

$$\tilde{\Phi} \rightarrow \Phi = \tfrac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t\phi - V(\phi - s). \tag{24}$$

We will also discuss briefly the exponential Lagrangian

$$\tilde{\mathcal{L}} \propto \exp\tilde{\Phi} \tag{25}$$

for a real field.

We will see that some of these models are related to Lagrangians of type (21) involving complex fields, that is, with $\tilde{\Phi}$ replaced by

$$\tilde{\Psi} \equiv \partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}^* - V(|\tilde{\psi}|). \tag{26}$$

For the complex fields we use equation (18) to find

$$\tilde{\Psi} \rightarrow \Psi = \nabla\psi \cdot \nabla\psi^* + \mathrm{i}a_0\frac{v_5}{v_4}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - V(|\psi|) \tag{27}$$

instead of (24). From (23) this reduces to

$$\tilde{\Psi} \rightarrow \Psi = \nabla\psi \cdot \nabla\psi^* - \mathrm{i}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - V(|\psi|). \tag{28}$$

We will see at the end of section 4.2 that it leads to the nonlinear Schrödinger equation.

2. Navier–Stokes equation

Let us consider the functional Lagrangian dependent on the two real fields $\tilde{\rho}$ and $\tilde{\phi}$:

$$\tilde{\mathcal{L}}[\tilde{\rho}, \tilde{\phi}] = -\tfrac{1}{2}\tilde{\rho}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - V(\tilde{\rho}). \tag{29}$$

The Euler–Lagrange equation for $\tilde{\rho}$ reduces to

$$\tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + V'(\tilde{\rho}) = 0 \tag{30}$$

where $V'(\tilde{\rho})$ is shorthand notation for the functional derivative $\frac{\delta}{\delta\tilde{\rho}}V(\tilde{\rho})$. Now consider the embedding given by equations (16) and (17) for the spacetime and the field $\tilde{\phi}$ as well as

$$\tilde{\rho}(x) \equiv \rho(\mathbf{x}, t). \tag{31}$$

Noting that

$$\begin{aligned} \partial_4\tilde{\phi} &= \frac{1}{v_4}\partial_t\phi \qquad & \partial_5\tilde{\phi} &= a_0 v_5 \\ \partial_4\tilde{\rho} &= \frac{1}{v_4}\partial_t\rho \qquad & \partial_5\tilde{\rho} &= 0 \end{aligned} \tag{32}$$

we find

$$\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} \to \nabla\phi\cdot\nabla\phi - 2a_0\frac{v_5}{v_4}\partial_t\phi. \tag{33}$$

From equation (23) we obtain the equation

$$\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi = -V'. \tag{34}$$

The gradient of this expression gives

$$(\nabla\phi\cdot\nabla)\nabla\phi + \partial_t(\nabla\phi) = -\nabla(V') \tag{35}$$

so that by identifying the field ϕ as a velocity potential, i.e. $\mathbf{v} \equiv \nabla\phi$ and the right-hand side as $-\frac{1}{\rho}\nabla p$ (where p is the pressure) we find the Navier–Stokes equation:

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p. \tag{36}$$

The identification of the right-hand side is valid for an isentropic fluid, where the pressure depends on the density ρ only. Then the force term $F = -V'$ can be understood as the enthalpy and $\sqrt{\rho V''}$ is the speed of sound.

The equation of motion with respect to the field $\tilde{\phi}$ is

$$\partial_\mu(\tilde{\rho}\partial^\mu\tilde{\phi}) = \partial_\mu\tilde{\rho}\partial^\mu\tilde{\phi} + \tilde{\rho}\partial_\mu\partial^\mu\tilde{\phi} = 0. \tag{37}$$

With the embedding described above it reduces to the continuity equation

$$\partial_t\rho + \nabla(\rho\nabla\phi) = 0. \tag{38}$$

The energy–momentum tensor

$$T^{\mu\nu} = g^{\mu\nu}\tilde{\mathcal{L}} - \partial^\mu\tilde{\phi}\frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_\nu\tilde{\phi})} \tag{39}$$

for the Lagrangian (29) is

$$T^{\mu\nu} = \tfrac{1}{2}g^{\mu\nu}\tilde{\rho}\partial_\alpha\tilde{\phi}\partial^\alpha\tilde{\phi} - g^{\mu\nu}V(\tilde{\rho}) - \tilde{\rho}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi}. \tag{40}$$

In terms of the embedding used earlier (and writing $v \equiv v_4 = v_5$) we find

$$\partial_\mu\partial^\mu\tilde{\phi} \to \nabla^2\phi \tag{41}$$

and the components of the energy–momentum tensor are

$$\begin{aligned}
&T^{jk} = \tfrac{1}{2}\delta_{jk}\rho\nabla^2\phi - \rho\partial^j\phi\partial^k\phi - \delta_{jk}V(\rho)\\
&T^{j4} = -v\rho\partial^j\phi \qquad T^{j5} = \frac{\rho}{v}\partial_j\phi\partial_t\phi\\
&T^{44} = -v^2\rho \qquad T^{55} = -\frac{\rho}{v^2}(\partial_t\phi)^2\\
&T^{45} = \rho\partial_t\phi - \tfrac{1}{2}\rho\nabla^2\phi + V(\rho).
\end{aligned} \tag{42}$$

Note that with the embedding given above, the Lagrangian in equation (29) becomes

$$\mathcal{L} = -\rho\left(\partial_t\phi + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi\right) - V(\rho). \tag{43}$$

As discussed in section 2 of [13], such a Lagrangian occurs in various contexts. In addition to the application to fluid motion we have just discussed, it describes (a) the continuous version of the free particle motion, where ρ is the density and ϕ first intervenes as a Lagrange multiplier which enforces the continuity equation and then can be identified as a velocity potential (see [23]); (b) the determinant of the induced metric in the Nambu–Goto action for a closed extended two-dimensional object moving in $3+1$ Minkowski spacetime as shown first by Bordemann and Hoppe [9], therein the variables t and s play the role of light-cone variables; (c) the Lagrangian of the Schrödinger equation from a hydrodynamical point of view [24].

In relation to the last application (hydrodynamical description of quantum mechanics) let us consider a Lagrangian of the type of equation (21):

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] = k_p \tilde{\Psi}^p \tag{44}$$

where $\tilde{\Psi}$ is given in equation (26), and show how the Navier–Stokes equation can be recovered for the case $p = 1$. (Other values of p are discussed in section 4.2.) Then equation (44) reads

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] = k_1(\partial_\mu \tilde{\psi} \partial^\mu \tilde{\psi}^* - V(|\tilde{\psi}|)) \tag{45}$$

and the Euler–Lagrange equation with respect to $\tilde{\psi}^*$ takes the form

$$\partial_\mu \partial^\mu \tilde{\psi} = -\frac{\delta V}{\delta \tilde{\psi}^*}. \tag{46}$$

If we define the real fields $\tilde{\rho}$ and $\tilde{\phi}$ by using the Madelung prescription [14, 24],

$$\tilde{\psi} \equiv \sqrt{\tilde{\rho}}\, \mathrm{e}^{\mathrm{i}\tilde{\phi}} \tag{47}$$

then we obtain the Lagrangian

$$\tilde{\mathcal{L}}[\tilde{\rho}, \tilde{\phi}] = k_1 \left(\tilde{\rho} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{1}{4\tilde{\rho}} \partial_\mu \tilde{\rho} \partial^\mu \tilde{\rho} - V(\sqrt{\tilde{\rho}}) \right) \tag{48}$$

(where the constant k_1 again has the units of velocity square) which can be written as

$$\tilde{\mathcal{L}} = k_1(\tilde{\rho} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \overline{V}(\tilde{\rho})) \tag{49}$$

where

$$\overline{V} \equiv \frac{1}{4\tilde{\rho}} \partial_\mu \tilde{\rho} \partial^\mu \tilde{\rho} - V. \tag{50}$$

Note that equation (49) coincides with (29) up to an arbitrary factor! The equation of motion with respect to $\tilde{\rho}$ is

$$\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{1}{4\tilde{\rho}^2} \partial_\mu \tilde{\rho} \partial^\mu \tilde{\rho} - \frac{1}{2\tilde{\rho}} \partial_\mu \partial^\mu \tilde{\rho} - V'(\tilde{\rho}) = 0. \tag{51}$$

Using the same embedding as usual we find the equation

$$\frac{1}{2} \nabla\phi \cdot \nabla\phi + \partial_t \phi = \frac{1}{2} V' + \frac{1}{4\rho} \nabla^2 \rho - \frac{1}{8\rho^2} \nabla\rho \cdot \nabla\rho \tag{52}$$

so that the gradient of the right-hand side is identified with $-\frac{1}{\rho}\nabla p$ in equation (36).

3. Chaplygin gas model

In this section we specialize the potential of the previous section to the Chaplygin gas model. Originally it was devised to describe the stationary flow in two dimensions of a compressible, isentropic gas without shock waves (see [25] for more details). It was noted in [9–11] that the non-relativistic Chaplygin gas model (related to d-branes and partons) also admits typically relativistic symmetries and, in the non-interacting case, conformal symmetries. The appearance of such symmetries is reminiscent of the hidden $O(4)$ symmetry of the hydrogen atom. In particular, the non-relativistic system discussed below admits a field-dependent diffeomorphism symmetry which coincides with the Poincaré Lie algebra in a space with one additional dimension [9]. Such models provide a nonlinear representation for the dynamical Poincaré group. Our interest in this Lagrangian is that this field-dependent

Poincaré symmetry was linearized by using a projection from a five-dimensional space such as the one used throughout this paper (see [14]). That different projections lead to either the non-relativistic Chaplygin model or to the relativistic Born–Infeld theory [14, 15], in our language, is equivalent to defining embeddings different from (16).

Let us restrict the potential in equation (29) to the expression

$$V(\tilde{\rho}) = \frac{\lambda}{\tilde{\rho}} \qquad \lambda > 0. \tag{53}$$

Therefore we start with a Lagrangian with two real fields $\tilde{\phi}$ and $\tilde{\rho}$:

$$\tilde{\mathcal{L}}[\tilde{\phi}, \tilde{\rho}] = -\frac{1}{2}\tilde{\rho}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{\lambda}{\rho}. \tag{54}$$

Here we omit the constant k_1. This Lagrangian describes the Chaplygin gas model, with enthalpy $V' = -\frac{\lambda}{\tilde{\rho}^2}$, negative pressure $p = -\frac{2\lambda}{\tilde{\rho}}$ and speed of sound $\frac{\sqrt{2\lambda}}{\tilde{\rho}}$. Initially Chaplygin introduced this potential in order to have an equation of state that would approximate the physically relevant adiabatic expressions with $V \propto \rho^\gamma, \gamma > 0$. This 'gas model' was used to describe some deformable bodies. Nowadays examples of negative pressure include the cosmological constant exerting negative pressure on the cosmos thereby accelerating expansion, exchange interactions in atoms and stripe states in the quantum Hall effect. Recent applications of the Chaplygin gas can be found in [26, 27] and its supersymmetric version has been worked out by Hassaïne in [28].

Using the embedding in equations (16) and (17) with conditions (23) as well as equation (31) the Lagrangian (54) reduces to

$$\mathcal{L} \propto -\rho\partial_t\phi - \frac{1}{2}\rho(\nabla\phi)^2 - V(\rho) \qquad \text{with} \quad V = \frac{\lambda}{\rho}. \tag{55}$$

The Euler–Lagrange equation with respect to the field $\tilde{\rho}$ gives

$$\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{\delta}{\delta\tilde{\rho}}V(\tilde{\rho}) = \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{\lambda}{\tilde{\rho}^2} = 0 \tag{56}$$

whereas for the field $\tilde{\phi}$ it yields

$$\partial_\mu(\tilde{\rho}\partial^\mu\tilde{\phi}) = \tilde{\rho}\partial_\mu\partial^\mu\tilde{\phi} + \partial_\mu\tilde{\rho}\partial^\mu\tilde{\phi} = 0. \tag{57}$$

The first equation reduces to the Bernoulli equation:

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = -\frac{\lambda}{\rho^2} \tag{58}$$

and the equation of motion for ϕ leads to the equation of continuity

$$\partial_t\rho + \nabla \cdot \mathbf{j} = 0 \tag{59}$$

where $\mathbf{j} = \rho\nabla\phi$. More details can be found in [15].

4. Models for superfluidity

At a temperature of 2.18 K liquid helium undergoes a thermodynamic transition and below this temperature it displays unusual properties one of which is superfluidity. Landau was the first to develop a quantitative description of the two-fluid aspects of helium II [25, 29]: the *superfluid* component is the part of the liquid that remains in the ground state, carries zero entropy and flows irrotationally, and the *normal* component, described by quasi-particles, behaves like any other viscous liquid. Landau proposed an energy versus momentum spectrum linear for small values of p:

$$E = cp \tag{60}$$

where c is the velocity of sound. This is referred to as phonon-like behaviour. Then the curve passes through a maximum, falls to a minimum before rising steeply for large values of the momentum. Near the minimum the spectrum is approximately described by the parabola

$$E = \Delta + \frac{(p - p_0)^2}{2\mu} \tag{61}$$

and is associated with rotons of effective mass μ and with Δ being the minimum energy required to create a roton at rest.

Some years later Kronig and Thellung suggested another formulation of quantum hydrodynamics based on a Lagrangian from which equations of motion may be derived and which allows one to use second quantization of the fields involved [30]. The main drawback of this model is that it is derived from a velocity potential so that the motion is always irrotational and therefore cannot describe roton excitations. The situation was rectified a year later by Thellung [31] and Ziman [32], and its main ingredient was the Clebsch parametrization of hydrodynamics (section 167 of [33]).

In sections 4.1 and 4.2 we describe models proposed by Takahashi for the irrotational components of the superfluid. The Lagrangian examined by Thellung and Ziman is discussed in section 4.3.

4.1. Barotropic irrotational fluid

In [1] we have used the Galilean metric formalism to express in a covariant way the Takahashi model for compressible irrotational barotropic fluids with pressure proportional to the square of the mass density. We then treated it like a classical field theory defined by a manifestly Galilei-covariant Lagrangian for a scalar field $\tilde{\phi}$ with a potential proportional to the square of the kinetic energy term, i.e. $(\partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi})^2$. Note that a potential depending on the derivatives also appears in equation (50). The specific Lagrangian is

$$\tilde{\mathcal{L}} = \frac{\rho_0}{8v_0^2} \left(\partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} - 2v_0^2\right)^2 \tag{62}$$

where ρ_0 and v_0^2 are constants that guarantee the coherence of units. The various dimensions are

$$\begin{aligned}
[\tilde{\mathcal{L}}] &= \frac{M}{LT^2} \qquad (M\text{: mass}) \\
[\partial] &= \frac{1}{L} \\
[\rho_0] &= \frac{M}{L^3} \\
[\tilde{\phi}] &= \frac{L^2}{T} \\
\left[v_0^2\right] &= [\partial\tilde{\phi}\partial\tilde{\phi}] = \frac{L^2}{T^2}.
\end{aligned} \tag{63}$$

The field $\tilde{\phi}$ is a Galilei scalar which after the embedding is related to a potential for the fluid velocity.

From the Lagrangian (62) the equation of motion for the field $\tilde{\phi}$ is

$$\partial_\mu \partial^\mu \tilde{\phi} - \frac{1}{2v_0^2}(\partial_\mu \partial^\mu \tilde{\phi})(\partial_\nu \tilde{\phi} \partial^\nu \tilde{\phi}) - \frac{1}{v_0^2}(\partial^\mu \tilde{\phi})(\partial^\nu \tilde{\phi})(\partial_\mu \partial_\nu \tilde{\phi}) = 0. \quad (64)$$

The corresponding five-dimensional energy–momentum tensor is

$$T^{\mu\nu} = \frac{\rho_0 g^{\mu\nu}}{8v_0^2}\left(\partial^\alpha \tilde{\phi}\partial_\alpha \tilde{\phi} - 2v_0^2\right)^2 - \frac{\rho_0}{2v_0^2}\partial^\mu \tilde{\phi}\partial^\nu \tilde{\phi}\left(\partial^\alpha \tilde{\phi}\partial_\alpha \tilde{\phi} - 2v_0^2\right) \quad (65)$$

where $g^{\mu\nu}$ is the Galilean metric (4). Its components and their physical interpretations are in [1]. Let us mention that equation (40) of that article makes it clear that the identity (1.1) used by Greiter *et al* [34] is automatically satisfied for a Lagrangian like (62). In their paper Greiter *et al* investigated the algebraic identity $J_k \propto T_{0k}$ in effective theories of superconductivity. Note also the similarity between equation (2.5) of Greiter *et al* and our equation (24). An equation similar to (2.13) of Greiter *et al* will be discussed in section 4.2.

In [1] it was shown that the embedding given in equations (16), (17) with (23) reduces the Lagrangian of equation (62) to

$$\mathcal{L} = \frac{\rho_0}{2v_0^2}\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t \phi - v_0^2\right)^2 \quad (66)$$

and the equation of motion becomes

$$v_0^2 \nabla^2 \phi - \partial_t^2 \phi = \nabla^2 \phi\left(\tfrac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t \phi\right) + \tfrac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi \cdot \nabla\phi) + 2\nabla(\partial_t \phi) \cdot \nabla\phi \quad (67)$$

which is equation (5.40) obtained by Takahashi [2]. Takahashi argued that the scalar field ϕ can be seen as a velocity potential of the (collective mode of the) fluid. He also claimed that if superfluidity is to be understood in terms of the spontaneous symmetry breaking of Galilean invariance, then this field is the best candidate for playing the role of a Goldstone boson. The phonon equation would then be replaced by equation (67).

Now let us express two other important equations of [2, 3] in a Galilei-covariant form. The first one is equation (6.5) of [2] which we now show to be obtained from the following Lagrangian with a complex field $\tilde{\psi}$, where we just replaced the derivative by a covariant-like derivative $D \equiv \partial + \mathrm{i}A$:

$$\tilde{\mathcal{L}} \propto D_\mu \tilde{\psi} D^\mu \tilde{\psi}^* \equiv (\partial_\mu \tilde{\psi} + \mathrm{i}\tilde{A}_\mu \tilde{\psi})(\partial^\mu \tilde{\psi}^* - \mathrm{i}\tilde{A}^\mu \tilde{\psi}^*) \quad (68)$$

where $\tilde{A}$ is similar to a gauge field defined by

$$\tilde{A}_\mu = k\partial_\mu \tilde{\phi} \quad (69)$$

for a real field $\tilde{\phi}$ for which the embedding will be defined as in equation (17). The constant k is determined below. Using the Euler–Lagrange equation for $\tilde{\psi}^*$ we find the equation of motion

$$\partial^\mu \partial_\mu \tilde{\psi} + \mathrm{i}(\partial^\mu \tilde{A}_\mu)\tilde{\psi} + 2\mathrm{i}\tilde{A}_\mu \partial^\mu \tilde{\psi} - \tilde{A}^\mu \tilde{A}_\mu \tilde{\psi} = 0. \quad (70)$$

With the embedding given by equations (16) and (18), as well as equation (69), where

$$\tilde{\phi}(x) \equiv \phi(\mathbf{x}, t) \quad (71)$$

then equation (70) reduces to

$$\nabla^2 \psi - 2\mathrm{i}a_0 \frac{v_5}{v_4}\partial_t \psi + \mathrm{i}k(\nabla \cdot \nabla\phi)\psi + 2\mathrm{i}k\nabla\phi \cdot \nabla\psi - 2\mathrm{i}k\frac{v_5}{v_4}\mathrm{i}a_0(\partial_t \phi)\psi - k^2(\nabla\phi \cdot \nabla\phi)\psi = 0. \quad (72)$$

Finally we choose $v_4 = v_5$, $a_0 = -m$ and $k = m$ to get

$$\mathrm{i}\partial_t \psi = m(\partial_t \phi)\psi - \frac{1}{2m}\left(\nabla + \mathrm{i}m\nabla\phi\right)^2 \psi \quad (73)$$

which is equation (6.5) of [2]. As explained by Takahashi [2], the field ψ describes the excitation of matter measured from the uniformly moving medium.

The second equation that we express here in a covariant form is equation (3.12) of [3]. This equation also involves a complex field $\tilde{\psi}$ as well as a real field $\tilde{\phi}$. First let us introduce a null vector

$$\tilde{u}_p^\mu \equiv \left(\mathbf{u}_p, v_4, \frac{1}{2v_5}\mathbf{u}_p^2\right). \tag{74}$$

Next, introduce the real scalar field $\tilde{\phi}$ with the embedding defined in equation (17).

Therefore we can write the scalar product

$$\tilde{\Phi}_p = \tilde{u}_p^\mu \partial_\mu \tilde{\phi} = \mathbf{u}_p \cdot \nabla\phi + \tfrac{1}{2}a_0\mathbf{u}_p^2 + \partial_t\phi \tag{75}$$

which upon using equation (23) gives

$$\Phi_p = \partial_t\phi + \mathbf{u}_p \cdot \nabla\phi - \tfrac{1}{2}\mathbf{u}_p^2. \tag{76}$$

Similarly construct

$$\Phi_m = \partial_t\phi + \mathbf{u}_m \cdot \nabla\phi - \tfrac{1}{2}\mathbf{u}_m^2 \tag{77}$$

where $\tilde{u}_p$ replaced with $\tilde{u}_m$. The subscripts p and m refer to the collective and individual modes of the fluid, respectively. The model presented below describes the interaction between the fluid as a whole and its constituents.

Now consider again the vector $\tilde{u}$, but contracted with the derivative of a complex scalar field $\tilde{\psi}$:

$$\begin{aligned}\chi_m &\equiv \frac{\mathrm{i}}{2}\tilde{u}_m^\mu[\tilde{\psi}^*\partial_\mu\tilde{\psi} - (\partial_\mu\tilde{\psi}^*)\tilde{\psi}] \\ &= \frac{\mathrm{i}}{2}\left(\psi^*[\partial_t\psi + \mathbf{u}_m\cdot\nabla\psi] - [\partial_t\psi^* + \mathbf{u}_m\cdot\nabla\psi^*]\psi + \mathrm{i}\mathbf{u}_m^2 a_0\psi^*\psi\right)\end{aligned} \tag{78}$$

where we have defined the embedding as in equations (16) and (18). If we choose $a_0 = 0$, which suggests that the field ψ is massless, then the last term in χ_m vanishes:

$$\chi_m = \psi^*[\partial_t\psi + \mathbf{u}_m\cdot\nabla\psi] - [\partial_t\psi^* + \mathbf{u}_m\cdot\nabla\psi^*]\psi. \tag{79}$$

Further defining

$$\eta_m(x) \equiv \alpha\psi^*(x) + \alpha^*\psi(x) \tag{80}$$

and

$$\rho_m(x) \equiv m\psi^*(x)\psi(x) \tag{81}$$

then the Lagrangian (3.12) of [3] takes the form

$$\mathcal{L} = \frac{\rho_0}{2v_0^2}\left(\Phi_p(x) - v_0^2\right)^2 + \eta_m(x)\Phi_p(x) + \rho_m(x)(\Phi_p(x) - \Phi_m(x)) + \chi_m(x) \tag{82}$$

where χ_m, η_m and ρ_m are given by equations (79), (80) and (81), respectively, Φ_p by equation (76) and Φ_m by (77). As discussed in [3], this is the simplest non-trivial Galilei invariant Lagrangian that could describe the interaction between the fluid as a whole (with velocity potential ϕ) and its constituents, the individual modes being denoted by the complex field ψ. The term η_m takes into account the mass transmutation between the p- and the m-modes. The mass density of the p-mode is defined by

$$\rho_p(x) \equiv \frac{\delta\mathcal{L}}{\delta\Phi_p} = -\frac{\rho_0}{v_0^2}\Phi_p(x) - \eta_m(x) - \rho_m(x) + \rho_0 \tag{83}$$

and the total mass density is

$$\rho(x) \equiv \rho_p(x) + \rho_m(x) = -\frac{\delta \mathcal{L}}{\delta(\partial_t \phi)} = -\frac{\rho_0}{v_0^2}\Phi_p(x) - \eta_m(x) + \rho_0. \tag{84}$$

The velocity field of the p-mode $\mathbf{u}_p$, determined by $\frac{\partial \mathcal{L}}{\partial \mathbf{u}_p} = \mathbf{0}$, is shown to be irrotational:

$$\mathbf{u}_p(x) = \nabla\phi(x). \tag{85}$$

For the m-mode the velocity $\mathbf{u}_m$, defined by $\frac{\partial \mathcal{L}}{\partial \mathbf{u}_p} = \mathbf{0}$, is such that

$$\rho_m(x)(\mathbf{u}_m(x) - \mathbf{u}_p(x)) = \frac{\hbar}{2\mathrm{i}}[\varphi^*(x)\nabla\varphi(x) - (\nabla\varphi^*(x))\varphi(x)] \tag{86}$$

which shows that the right-hand side is the mass current of the m-mode relative to the velocity of the p-mode.

4.2. Generalized models for non-barotropic fluids

In this section we generalize the Lagrangian of equation (62) by relaxing the barotropic condition

$$p \propto \rho^2 \tag{87}$$

where p is the pressure and ρ its density, as suggested by Takahashi. In section 2 of [3] it is proposed to replace the condition (87) with

$$p \propto \rho^\gamma \qquad \gamma > 1 \tag{88}$$

which amounts to choosing the Lagrangian as in equation (21). Another suggestion is to have

$$p \propto \rho \tag{89}$$

which is equivalent to choosing the Lagrangian as in equation (25).

Let us consider $\tilde{\Phi}$ as in equation (22). Then the potential must be independent of $\tilde{\phi}$, that is

$$\tilde{\Phi} \equiv \tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - v_0^2 \tag{90}$$

as in equation (62) otherwise the extra parameter s remains in the equations of motion after the embedding has been defined. Let us consider first the Lagrangian of equation (21),

$$\tilde{\mathcal{L}}[\tilde{\phi}] = k_p\tilde{\Phi}^p \tag{91}$$

where the units of the constant k_p are

$$[k_p] = MT^{p-2}L^{-p-1} \tag{92}$$

to ensure that $\tilde{\mathcal{L}}$ has the correct dimensions. This corresponds to equation (62) with the particular values

$$p = 2 \qquad \text{and} \qquad k_2 = \frac{\rho_0}{2v_0^2}. \tag{93}$$

By using the Euler–Lagrange equation we find the equation of motion:

$$\left(\tfrac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - v_0^2\right)\partial_\nu\partial^\nu\tilde{\phi} + (p-1)\partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0. \tag{94}$$

This reduces to equation (64) with conditions (93). With the embedding of equations (16) and (23) we find from equation (24) that

$$\tilde{\Phi} \to \Phi = \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi - v_0^2 \tag{95}$$

and from equation (94) we see that equation (67) is generalized to

$$v_0^2\nabla^2\phi - (p-1)\partial_t^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right) + (p-1)\nabla\phi\cdot\nabla\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + 2\partial_t\phi\right). \quad (96)$$

Note that for $p = 1$ this equation is simplified to

$$v_0^2\nabla^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right). \quad (97)$$

For $p \neq 1$ the general equation (96) is equivalent to the Takahashi model.

Another suggestion made by Takahashi in [3] is the exponential model, equation (25), based on the Lagrangian

$$\tilde{\mathcal{L}}[\tilde{\phi}] = k\exp\left(\frac{\tilde{\Phi}}{v_0^2}\right) \quad (98)$$

where $\tilde{\Phi}$ is given by equation (90). The Euler–Lagrange equation gives

$$v_0^2\partial_\mu\partial^\mu\tilde{\phi} + \partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0 \quad (99)$$

before any embedding has been performed. The embedding in equation (16) together with (23) leads to the equation

$$v_0^2\nabla^2\phi + \partial_{tt}\phi + \nabla\phi\cdot\nabla\left(\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + 2\partial_t\phi\right) = 0. \quad (100)$$

Now we obtain some equations relevant to superconductivity by considering Lagrangians of the type given in equation (21) with various choices of the potential V for *complex* fields, that is, equation (44) with $\tilde{\Psi}$ given in equation (26):

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] = k_p(\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}^* - V(|\tilde{\psi}|))^p. \quad (101)$$

Note that when

$$V(|\tilde{\psi}|) = \lambda|\tilde{\psi}|^{2N} \quad (102)$$

then

$$\frac{\delta V}{\delta\tilde{\psi}^*} = N\lambda\tilde{\psi}|\tilde{\psi}|^{2N-2}. \quad (103)$$

By using the Euler–Lagrange equation for ψ^*, we find the equation of motion:

$$(p-1)\left(\partial_{\mu\nu}\tilde{\psi}\partial^\mu\tilde{\psi}\partial^\nu\tilde{\psi}^* + \partial_{\mu\nu}\tilde{\psi}^*\partial^\mu\tilde{\psi}\partial^\nu\tilde{\psi} - \frac{\delta V}{\delta\tilde{\psi}}\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi} - \frac{\delta V}{\delta\tilde{\psi}^*}\partial_\mu\tilde{\psi}^*\partial^\mu\tilde{\psi}\right)$$
$$+ (\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}^* - V(|\tilde{\psi}|))\left(\partial_\nu\partial^\nu\tilde{\psi} + \frac{\delta V}{\delta\tilde{\psi}^*}\right) = 0. \quad (104)$$

This equation with $V = v_0^2$ reduces to

$$\partial_\mu\partial^\mu\tilde{\psi}\left(\partial_\nu\tilde{\psi}\partial^\nu\tilde{\psi}^* - v_0^2\right) + (p-1)(\partial_{\mu\nu}\tilde{\psi}\partial^\nu\tilde{\psi}^* + \text{h.c.})\partial^\mu\tilde{\psi} = 0 \quad (105)$$

where 'h.c.' is the Hermitian conjugate of the first term.

In the following we define the complex field as in equation (18) together with the spacetime embedding given by equations (16) so that $\tilde{\Psi}$ reduces to equation (27). Let us consider the power $p = 1$ so that equation (104) becomes

$$\partial_\mu\partial^\mu\tilde{\psi} - \frac{\delta V}{\delta\tilde{\psi}^*} = 0 \quad (106)$$

as in equation (46). For the potential given in equation (102) this becomes

$$\partial_\mu \partial^\mu \tilde{\psi} + N\lambda \tilde{\psi} |\tilde{\psi}|^{2N-2} = 0. \tag{107}$$

For example, the choice $p = 1$ and $V = \lambda |\tilde{\psi}|^4$ corresponds to the model considered in [34]. If we define the real fields $\tilde{\rho}$ and $\tilde{\phi}$ using Madelung's transcription, equation (47), then we obtain from equation (48):

$$\tilde{\mathcal{L}} = k_1 \left(\tilde{\rho} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{1}{4\tilde{\rho}} \partial_\mu \tilde{\rho} \partial^\mu \tilde{\rho} - \lambda \tilde{\rho}^2 \right) \tag{108}$$

which is the covariant expression for equation (5) of Liu's article [34]. For this quartic potential we have $N = 2$ and the Lagrangian (101) becomes

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] = k_1 (\partial_\mu \tilde{\psi} \partial^\mu \tilde{\psi}^* - \lambda |\tilde{\psi}|^4). \tag{109}$$

Using the embedding in equations (16) and (18) we find the Lagrangian

$$\mathcal{L} = k_1 (\nabla \psi \cdot \nabla \psi^* - \mathrm{i} m (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \lambda |\psi|^4). \tag{110}$$

The corresponding equation (107) is just

$$\partial_\mu \partial^\mu \tilde{\psi} + 2\lambda |\tilde{\psi}|^2 \tilde{\psi} = 0. \tag{111}$$

From the embedding in equations (16) and (18) we find

$$\nabla^2 \psi - 2\mathrm{i} a_0 \frac{v_5}{v_4} \partial_t \psi - 2\lambda |\psi|^2 \psi = 0. \tag{112}$$

With $v_4 = v_5 = v$ and $a_0 = -m$ we obtain the well-known nonlinear Schrödinger equation

$$\mathrm{i} \partial_t \psi = -\frac{1}{2m} \nabla^2 \psi + \frac{\lambda}{m} |\psi|^2 \psi. \tag{113}$$

A similar equation is used in [34].

To conclude this section, let us mention that in [35] a similar non-relativistic Lagrangian is obtained in a different manner. The author starts with the relativistic *four-dimensional* Lorentz-covariant Lagrangian:

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \chi \partial^\mu \chi - \tfrac{1}{2} m^2 \chi^2 - \tfrac{1}{4!} \lambda \chi^4 - \tfrac{1}{2} \delta m^2 \chi^2 \tag{114}$$

for a real field χ, and introduces the *non-relativistic* fields ψ and ψ^* by performing the substitution

$$\chi \equiv \frac{1}{\sqrt{2m}} (\mathrm{e}^{-\mathrm{i} m t} \psi + \mathrm{e}^{\mathrm{i} m t} \psi^*). \tag{115}$$

The Lagrangian density obtained thereby is reduced to equation (110).

4.3. Model of non-viscous fluids and liquid helium

Unlike the previous sections the model described here allows rotational motion in the superfluid. We start with a covariant Lagrangian similar to equation (29) but where we perform a five-dimensional Clebsch-like transformation of the velocity field

$$\partial \tilde{\phi} \to \partial \tilde{\phi} + \tilde{\alpha} \partial \tilde{\beta} \tag{116}$$

so that

$$\tilde{\mathcal{L}} = -\frac{\tilde{\rho}}{2v_0^2} (\partial_\mu \tilde{\phi} + \tilde{\alpha} \partial_\mu \tilde{\beta})(\partial^\mu \tilde{\phi} + \tilde{\alpha} \partial^\mu \tilde{\beta}) - V(\tilde{\rho}) \tag{117}$$

where v_0 is a constant to ensure the coherence of units. For the four fields involved in equation (117) we use the embeddings

$$\tilde{\alpha}(x) = \alpha(\mathbf{x}, t) \qquad \tilde{\beta}(x) = \beta(\mathbf{x}, t) \qquad \tilde{\rho}(x) = \rho(\mathbf{x}, t) \tag{118}$$

as well as equation (17) for $\tilde{\phi}(x)$, and equation (16) for spacetime. Note that we shall choose $v_4 = v_5$ as in equation (23) but here we take $a_0 = +1$. Then the Lagrangian of equation (117) after projection onto the Newtonian spacetime becomes

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi - \frac{1}{2}\nabla\phi\cdot\nabla\phi + \alpha\left(\partial_t\beta - \frac{1}{2}\alpha\nabla\beta\cdot\nabla\beta\right) - \alpha\nabla\phi\cdot\nabla\beta\right) - V(\rho). \tag{119}$$

This may be brought to the form used earlier [31, 32]:

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi + \alpha\partial_t\beta - \frac{1}{2}\mathbf{v}^2\right) - V(\rho) \tag{120}$$

where we define the velocity vector as

$$\mathbf{v} \equiv -\nabla\phi - \alpha\nabla\beta. \tag{121}$$

The Euler–Lagrange equation for the field $\tilde{\rho}$ with equation (117) is

$$\frac{1}{2v_0^2}(\partial_\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\beta})(\partial^\mu\tilde{\phi} + \tilde{\alpha}\partial^\mu\tilde{\beta}) + V'(\tilde{\rho}) = 0 \tag{122}$$

and with the embedding described above, it projects to

$$\partial_t\phi + \alpha\partial_t\beta - \tfrac{1}{2}\mathbf{v}^2 = -v_0^2 V'(\rho) \tag{123}$$

with $\mathbf{v}$ given by equation (121). The equation of motion with respect to the field $\tilde{\phi}$ in five dimensions is similar to a continuity equation

$$\partial_\mu[\tilde{\rho}(\partial^\mu\tilde{\phi} + 2\tilde{\alpha}\partial^\mu\tilde{\beta})] = 0 \tag{124}$$

with 5-current $J^\mu = \tilde{\rho}(\partial^\mu\tilde{\phi} + 2\tilde{\alpha}\partial^\mu\tilde{\beta})$. In Newtonian spacetime this equation reduces to

$$\partial_t\rho - \nabla\rho\cdot(\nabla\phi + 2\alpha\nabla\beta) - \rho(\nabla^2\phi + 2\nabla\alpha\cdot\nabla\beta + \nabla^2\beta) = 0. \tag{125}$$

For the field $\tilde{\alpha}$ the five-dimensional equation is

$$(\partial_\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\beta})\partial^\mu\tilde{\beta} = 0 \tag{126}$$

and with the same embedding as before, it becomes

$$\partial_t\beta + \mathbf{v}\cdot\nabla\beta = 0 \tag{127}$$

in Newtonian spacetime. Finally when we calculate the Euler–Lagrange equation for the field $\tilde{\beta}$, we obtain

$$\tilde{\alpha}\partial_\mu\partial^\mu\tilde{\phi} + \partial_\mu\tilde{\alpha}\partial^\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\alpha}\partial^\mu\tilde{\beta} + \tfrac{1}{2}\tilde{\alpha}^2\partial_\mu\partial^\mu\tilde{\beta} = 0 \tag{128}$$

and its four-dimensional version is

$$\partial_t\alpha - \alpha\nabla^2\phi - \tfrac{1}{2}\alpha^2\nabla^2\beta + \mathbf{v}\cdot\nabla\alpha = 0. \tag{129}$$

Let us now turn to the five-dimensional energy–momentum tensor, equation (39), which is for the Lagrangian in equation (117):

$$T^{\mu\nu} = \frac{\tilde{\rho}}{2v_0^2}[\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} + 2\tilde{\alpha}(\partial^\mu\tilde{\phi}\partial^\nu\tilde{\beta} + \partial^\mu\tilde{\beta}\partial^\nu\tilde{\phi}) + \tilde{\alpha}^2\partial^\mu\tilde{\beta}\partial^\nu\tilde{\beta} - g^{\mu\nu}(\partial\tilde{\phi} + \tilde{\alpha}\partial\tilde{\beta})^2]. \tag{130}$$

The embedding employed in this section implies that

$$(\partial\tilde{\phi} + \tilde{\alpha}\partial\tilde{\beta})^2 \to -2\partial_t\phi + \nabla\phi\cdot\nabla\phi + 2\alpha\nabla\phi\cdot\nabla\beta - 2\alpha\partial_t\beta + \alpha^2\nabla\beta\cdot\nabla\beta. \tag{131}$$

The components of the tensor are

$$\begin{aligned}
T^{jk} &= \frac{\rho}{2v_0^2}[\partial_j\phi\partial_k\phi + 2\alpha(\partial_j\phi\partial_k\beta + \partial_k\phi\partial_j\beta) + \alpha^2\partial_j\beta\partial_k\beta \\
&\qquad - \delta_{jk}(-2\partial_t\phi + \nabla\phi\cdot\nabla\phi + 2\alpha\nabla\phi\cdot\nabla\beta - 2\alpha\partial_t\beta + \alpha^2\nabla\beta\cdot\nabla\beta)] \\
T^{j4} &= -\frac{v_5\rho}{2v_0^2}(\partial_j\phi + 2\alpha\partial_j\beta) \\
T^{j5} &= -\frac{\rho}{2v_0^2v_4}[\partial_j\phi\partial_t\phi + 2\alpha(\partial_j\phi\partial_t\beta + \partial_j\beta\partial_t\phi) + \alpha^2\partial_j\beta\partial_t\beta] \\
T^{44} &= \frac{\rho v_5}{2v_0^2} \\
T^{55} &= \frac{\rho}{2v_0^2v_4^2}[(\partial_t\phi)^2 + 4\alpha\partial_t\phi\partial_t\beta + \alpha^2(\partial_t\beta)^2] \\
T^{45} &= \frac{\rho}{2v_0^2}[\partial_t\phi + 2\alpha\partial_t\beta - 2\partial_t\phi + \nabla\phi\cdot\nabla\phi + 2\alpha\nabla\phi\cdot\nabla\beta - 2\alpha\partial_t\beta + \alpha^2\nabla\beta\cdot\nabla\beta].
\end{aligned} \tag{132}$$

5. Concluding remarks

The general scheme followed in this paper (and emphasized in [2, 3, 15] among others) is to use the concepts of symmetry and geometry in quantum field theory to establish some prescription for writing Lagrangians for non-relativistic hydrodynamics. Specifically, the metric formulation of Galilei invariance has been utilized to express many fluids models in a covariant form, in much the same way as Lorentz covariance is usually a criterion for building action functionals for relativistic systems. Let us emphasize again that most Lagrangians discussed here consist of rather simple modifications of the Galilei-invariant $(\partial\phi)^2$. After recovering these models the next step is to use this formalism to propose new Lagrangians. An obvious example would consist in mixing various expressions discussed here, for instance, by substituting the Clebsch transformation (116) into Lagrangians of section 4 so as to obtain the corresponding equations with non-vanishing vorticity. Obviously other invariants could be considered, such as in higher spin theories. We are currently investigating the Fokker–Planck equation as a gauge field using the current setting; in particular, a non-Abelian generalization appears in a natural way.

An important aspect of this metric formalism which needs to be examined is its quantization. We are currently investigating it for a scalar field. In the present context this would allow a deeper study of quantum fluids. For instance, one could obtain five-dimensional versions of the spectrum of energy versus momentum of liquid helium, given in equations (60) and (61).

As a group-theoretical application, it might be interesting to extend the present technique to other Lie groups related by contraction, such as the de Sitter and Galilei groups, both omnipresent in this paper. An argument in favour of such an approach is that the five-dimensional formalism of Galilean covariance is due to the fact that the Galilei group admits a non-trivial central charge [4] and the contraction procedure often has exactly this property of generating Lie algebras with central extension [22].

Acknowledgments

We are grateful to NSERC (Canada) and to CNPq (Brazil) for financial support.

References

[1] de Montigny M, Khanna F C and Santana A E 2001 On Galilei-covariant Lagrangian models of fluids *J. Phys. A: Math. Gen.* **34** 10921–37

[2] Takahashi Y 1988 Towards the many-body theory with the Galilei invariance as a guide. I *Fortschr. Phys.* **36** 63–81

[3] Takahashi Y 1988 Towards the many-body theory with the Galilei invariance as a guide. II *Fortschr. Phys.* **36** 83–96

[4] Omote M, Kamefuchi S, Takahashi Y and Ohnuki Y 1989 Galilean covariance and the Schrödinger equation *Fortschr. Phys.* **37** 933–50

[5] Santana A E, Khanna F C and Takahashi Y 1998 Galilei covariance and (4, 1)-de Sitter space *Prog. Theor. Phys.* **99** 327–36

de Montigny M, Khanna F C, Santana A E, Santos E S and Vianna J D M 1999 Poincaré gauge theory and Galilean covariance *Ann. Phys., NY* **277** 144–58

de Montigny M, Khanna F C and Santana A E 2001 Galilean covariance and applications in physics *Recent Research Developments in Physics* vol 1 (Trivandrum: Transworld Reasearch Network) 45–92

[6] Pinski G 1968 Galilean tensor calculus *J. Math. Phys.* **9** 1927–30

[7] Soper D E 1976 *Classical Field Theory* (New York: Wiley) section 7.3

[8] Duval C, Burdet G, Künzle H P and Perrin M 1985 Bargmann structures and Newton–Cartan theory *Phys. Rev.* D **31** 1841–53

Künzle H P and Duval C 1994 Relativistic and nonrelativistic physical theories on five-dimensional space-time *Semantical Aspects of Spacetime Theories* ed U Majer and H J Schmidt (Mannheim: BI-Wissenschaftsverlag) pp 113–29

[9] Bazeia D and Jackiw R 1988 Nonlinear realization of a dynamical Poincaré symmetry by a field-dependent diffeomorphism *Ann. Phys., NY* **270** 246–59

[10] Bordemann M and Hoppe J 1993 The dynamics of relativistic membranes. Reduction to 2-dimensional fluid dynamics *Phys. Lett.* B **317** 315–20

[11] Jevicki A 1998 Light-front partons and dimensional reduction in relativistic field theory *Phys. Rev.* D **57** R5955–8

[12] Jackiw R and Polychronakos A P 1999 Fluid dynamical profiles and constants of motion from d-branes *Commun. Math. Phys.* **207** 107–29

[13] Jackiw R and Polychronakos A P 1999 Dynamical Poincaré symmetry realized by field-dependent diffeomorphisms *Proc. Steklov Inst. Math.* **226** 193 (*Preprint* hep-th/9809123) (this preprint is based on lectures delivered at Faddeev Festschrift, Steklov Mathematical Institute)

[14] Hassaïne M and Horváthy P A 2000 Field-dependent symmetries of a non-relativistic fluid model *Ann. Phys., NY* **282** 218–46

Hassaïne M and Horváthy P A 2001 Relativistic Chaplygin gas with field-dependent Poincaré symmetry *Lett. Math. Phys.* A **57** 33–40

[15] Jackiw R 2002 *Lectures on Fluid Dynamics—A Particle Theorist's View of Supersymmetric, Non-Abelian, Noncommutative Fluid Mechanics and d-Branes (CRM Series in Mathematical Physics)* (New York: Springer)

[16] Hassaïne M and Horváthy P A 2001 Symmetries of fluid dynamics with polytropic exponent *Phys. Lett.* A **279** 215–22

Hassaïne M and Horváthy P A 2000 The symmetries of the Manton superconductivity model *J. Geom. Phys.* **34** 242–62

[17] Hassaïne M, Horváthy P A and Yera J C 1998 Non-relativistic Maxwell–Chern–Simons vortices *Ann. Phys., NY* **263** 276–94

[18] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 Observation of a Bose–Einstein condensation in a dilute atomic vapour *Science* **269** 198–201

Davis K B, Mewes M O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 Bose–Einstein condensation in a gas of sodium atoms *Phys. Rev. Lett.* **75** 3969–73

Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions *Phys. Rev. Lett.* **75** 1687–90

[19] Fried D G, Killian T C, Willmann L, Landhuis D, Moss S C, Kleppner O and Greytak T J 1998 Bose–Einstein condensation of atomic hydrogen *Phys. Rev. Lett.* **81** 3811–4

[20] Raman C, Köhl M, Onofrio R, Durfee D S, Kuklewicz C E, Hadzibabic Z and Ketterle W 1999 Evidence for a critical velocity in a Bose–Einstein condensed gas *Phys. Rev. Lett.* **83** 2502–5

Madison K W, Chevy F, Wohlleben W and Dalibard J 2000 Vortex formation in stirred Bose–Einstein condensate *Phys. Rev. Lett.* **84** 806–9

[21] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Theory of Bose–Einstein condensation in trapped gases *Rev. Mod. Phys.* **71** 463–512

[22] de Azcárraga J A and Izquierdo J M 1995 *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics* (New York: Cambridge University Press)

[23] Eckart C 1938 The electrodynamics of material media *Phys. Rep.* **54** 920–3

[24] Madelung E 1927 Quantentheorie in hydrodynamischer form *Z. Phys.* **40** 322–6

[25] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (New York: Pergamon)

[26] Fabris J C, Gonçalves S V B and de Souza P E 2002 Mass power spectrum in a Universe dominated by the Chaplygin gas *Gen. Relat. Grav.* **34** 2111–26

Benaoum H B 2002 Accelerated universe from modified Chaplygin gas and tachyonic fluid *Preprint* hep-th/0205140

Fabris J C, Gonçalves S V B and de Souza P E 2002 Fitting the supernova Type 1a data with the Chaplygin gas *Preprint* astro-ph/0207430

[27] Bilic N, Tupper G B and Viollier R D 2002 Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas *Phys. Lett.* B **535** 17–21

[28] Hassaïne M 2001 Supersymmetric Chaplygin gas *Phys. Lett.* A **290** 157–64

[29] Landau L D 1941 The theory of superfluidity of Helium II *J. Phys. USSR* **5** 71–90

Landau L D 1947 On the theory of superfluidity of Helium II *J. Phys. USSR* **11** 91–2

The two previous papers are reprinted for convenience in: Khalatnikov I M 1965 *An Introduction to the Theory of Superfluidity* (New York: Benjamin)

[30] Kronig R and Thellung A 1952 On the hydrodynamics of non-viscous fluids and the theory of helium II *Physica* **18** 749–61

[31] Thellung A 1953 On the hydrodynamics of non-viscous fluids and the theory of helium II, Part II *Physica* **19** 217–26

[32] Ziman J M 1953 Quantum hydrodynamics and the theory of liquid helium *Proc. R. Soc.* A **219** 257–70

[33] Lamb H 1997 *Hydrodynamics* (Cambridge: Cambridge University Press)

[34] Greiter M, Wilczek F and Witten E 1989 Hydrodynamics relations in superconductivity *Mod. Phys. Lett.* B **3** 903–18

Liu W V 1998 Effective field theory approach to Bose–Einstein condensation *Int. J. Mod. Phys.* B **12** 2103–14

[35] Bergman O 1992 Nonrelativistic field-theoretic scale anomaly *Phys. Rev.* D **46** 5474–8